Lecture 21

May 4th, 2004

Higher Elliptic Regularity

We may obtain using our previous result yet higher regularity. This is a key point in regularity theory, and is quite beautiful. The previous theorem guaranteed that under certain conditions on the coefficients any $W^{1,2}(\Omega)$ solution is in fact $W^{2,2}(\Omega')$ for any $\Omega' \Subset \Omega$. Naturally we would expect that if we had assumed more regularity on the coefficients we would get that any $W^{2,2}(\Omega)$ solution is in fact $W^{3,2}(\Omega')$. If this could work then we could thus start from a $W^{1,2}$ solution, a weak solution, and get arbitrary regularity if we are willing to shrink more and more the domain. Alternatively we could get just enough regularity so that our embedding theorems ensure us the solution is Hölder , then $\mathcal{C}^{1,\alpha}$, $\mathcal{C}^{2,\alpha}$ and by repeating this process any desired regularity! (Here we use the Corollary of Lecture 18) This is really something. We started from a weak solution which need not be a function and using our theory we are able to show it behaves well and is in fact smooth! We will make this discussion precise in the sequel.

Let us see how this interpolation process works.

We assume as before $u \in W^{1,2}(\Omega)$ is a weak solution of $Lu = f$ (though we know this implies more regularity in the interior, we won't use it now). Then

$$\forall \; v \in \mathcal{C}_0^1(\Omega) \qquad -\int_\Omega a^{ij}\mathrm{D}_j u \mathrm{D}_i v + \int_\Omega (b^i \mathrm{D}_i u + cu)v = \int_\Omega fv,$$

and the idea is now to look at a smaller space of test functions v and see what that gives. In a sense the extra regularity we find will come for free. Take in particular $v \in \mathcal{C}_0^2(\Omega)$. That means $v = \mathrm{D}_k w$ where $w \in \mathcal{C}_0^1(\Omega)$, and

$$\int_\Omega a^{ij}\mathrm{D}_j u \mathrm{D}_i \mathrm{D}_k w = \int_\Omega (-b^i \mathrm{D}_i u - c + f)\mathrm{D}_k w.$$

Since w is twice continuously differentiable we may interchange derivatives above and integration by parts yields

$$-\int_\Omega \mathrm{D}_k(a^{ij}\mathrm{D}_j u)\mathrm{D}_i w = \int_\Omega (-b^i\mathrm{D}_i u - c + f)\mathrm{D}_k w,$$

and

$$-\int_\Omega a^{ij}\mathrm{D}_k\mathrm{D}_j u\mathrm{D}_i w = \int_\Omega \mathrm{D}_k a^{ij}\mathrm{D}_j u\mathrm{D}_i w + \int_\Omega (-b^i\mathrm{D}_i u - c + f)\mathrm{D}_k w,$$

and further

$$\int_\Omega a^{ij}\mathrm{D}_j(\mathrm{D}_k u)\mathrm{D}_i w = \int_\Omega [-\mathrm{D}_i(\mathrm{D}_k a^{ij}\mathrm{D}_j u) + \mathrm{D}_k(b^i\mathrm{D}_i u - c + f)]\cdot w =: \int_\Omega g\cdot w,$$

which gives that $\mathrm{D}_k u$ is a weak solution of the second order equation

$$\tilde{L}(\mathrm{D}_k u) = g$$

since this holds $\forall\ w \in \mathcal{C}_0^1(\Omega)$.

Now we note that $\tilde{L} = a^{ij}$ is strictly elliptic, and that if

- $a^{ij} \in \mathcal{C}^{1,1}(\Omega)$
- $b^i, c \in \mathcal{C}^{0,1}(\Omega)$
- $u \in W^{2,2}(\Omega)$

then we will have $g \in L^2(\Omega)$, and the Theorem of the previous lecture applies and we have $\mathrm{D}_k u \in W^{2,2}(\Omega') \quad \forall\ \ \Omega' \Subset \Omega$, i.e $u \in W^{3,2}(\Omega')$ as we wished to show. Indeed we get this extra regularity seemingly for free and we may continue this for higher derivatives.

Let us see what kind of a priori estimates we get on the higher norms. From the Theorem we have

$$\begin{aligned}||\mathrm{D}_k u||_{W^{2,2}(\Omega')} &\leq c\big(||\mathrm{D}_k u||_{W^{1,2}(\Omega)} + ||g||_{L^2(\Omega)}\big).\\ &\leq c\big(||u||_{W^{2,2}(\Omega)} + ||u||_{W^{1,2}(\Omega)} + ||u||_{W^{2,2}(\Omega)} + ||f||_{L^2(\Omega)}\big)\end{aligned}$$

where the last three terms come from the definition of g. We now shrink from Ω to Ω' and from Ω' to Ω'' so that terms on the LHS are evaluated on Ω'' and the ones on the RHS on Ω'. But then those terms on the RHS can be evaluated by Ω terms using our theorem once again! We get altogether then

$$||\mathrm{D}_k u||_{W^{2,2}(\Omega'')} \leq c\big(||u||_{W^{1,2}(\Omega)} + ||f||_{L^2(\Omega)}\big).$$

We state this as the following

Theorem. *Let $u \in W_0^{1,2}(\Omega)$ be a weak solution of $Lu = f$ in Ω, and assume*

- *L strictly elliptic with $(a^{ij}) > \gamma \cdot I, \ \ \gamma > 0$*
- *$a^{ij} \in \mathcal{C}^{k,1}(\bar{\Omega})$*
- *$b^i, c \in \mathcal{C}^{k-1,1}(\bar{\Omega})$*
- *$f \in W^{k,2}(\Omega)$.*

Then for any precompact set $\Omega' \Subset \Omega, \ \ u \in W^{k+2,2}(\Omega')$ and

$$||u||_{W^{2,2}(\Omega')} \leq C(||a^{ij}||_{C^{k,1}(\Omega)}, ||b||_{C^{k-1,1}(\Omega)}, ||c||_{C^{k-1,1}(\Omega)}, \lambda, \Omega', \Omega, k, n) \cdot \big(||u||_{W^{1,2}(\Omega)} + ||f||_{W^{k,2}(\Omega)}\big).$$

What we just did is the analogue in regularity theory of the technique we used in the Hölder part of the course. As there, we want to differentiate the original equation but the lack of regularity hinders us from doing do directly. We then take difference quotients and get bounds which now allow us to differentiate and get all higher estimates. This shows us how special solutions of such partial differetial equations are among general functions in those Sobolev Spaces.

Corollary. *Let $u \in W_0^{1,2}(\Omega)$ be a weak solution of $Lu = f$ in Ω, and assume*

- *L strictly elliptic with $(a^{ij}) > \gamma \cdot I, \ \ \gamma > 0$*
- *$f, a^{ij}, b^i, c \in \mathcal{C}^{\infty}(\bar{\Omega})$*

Then on the whole domain, $\ u \in \mathcal{C}^{\infty}(\Omega)$.

Proof. For all $k \in \mathbb{N}$, $f \in W^{k,2}(\Omega) \quad \Rightarrow \quad \Omega' \Subset \Omega \;\; u \in W^{k+2,2}(\Omega')$. By the Sobolev Embedding (Corollary to Lecture 18) then $u \in \mathcal{C}^m(\Omega'), m < k+2-\frac{n}{2}$, hence $u \in \mathcal{C}^\infty(\Omega')$. Apply this reasoning for Ω' a ball around each point $p \in \Omega$! to get $u \in \mathcal{C}^\infty(\Omega)$. Smoothness is indeed a notion defined pointwise. ■

Global Regularity (upto the boundary)

$\mathcal{U}$p until now our regularity results were for the space $W_0^{1,2}(\Omega)$, i.e functions which vanish on $\partial\Omega$. We now study $W^{1,2}(\Omega)$.

Theorem. *Suppose $u \in W^{1,2}(\Omega)$ is a (weak) solution of $Lu = f$ and assume*

- *L strictly elliptic with $(a^{ij}) > \gamma \cdot I, \;\; \gamma > 0$*
- $a^{ij} \in \mathcal{C}^{0,1}(\bar{\Omega})$
- $b^i, c \in L^\infty(\bar{\Omega})$
- $f \in L^2(\Omega)$
- *Ω has $\mathcal{C}^2$ boundary.*
- *$\exists\; \varphi \in W^{2,2}(\Omega)$ such that $u - \varphi \in W_0^{1,2}(\Omega)$.*

Then $u \in W^{2,2}(\Omega) = W^{2,2}(\bar{\Omega})$ on all of Ω with

$$||u||_{W^{2,2}(\Omega')} \leq C(||a^{ij}||_{C^{0,1}(\Omega)}, ||b||_{C^0(\Omega)}, ||c||_{C^0(\Omega)}, \lambda, \Omega', \Omega, \partial\Omega, n) \cdot$$

$$\cdot \big(||u||_{W^{1,2}(\Omega)} + ||f||_{L^2(\Omega)} + ||\varphi||_{W^{2,2}(\Omega)}\big).$$

Note that even for the 0 boundary values case this theorem gives a stronger conclusion: regularity upto the boundary with a uniform estimate on all of Ω. The price is the assumption that the boundary is regular enough.

Proof. We reduce to the zero boundary case in the usual manner. Suppose we could prove the Theorem for all zero boundary Dirichlet Problems. Then for the problem $L(u-\varphi) = f' := f - L\varphi$ on Ω, $u - \varphi = 0$ on $\partial\Omega$ (this is precisely the assumption $u - \varphi \in W_0^{1,2}(\Omega)$) we would have the desired estimates

$$||u-\varphi||_{W^{2,2}(\Omega')} \leq C \cdot \big(||u-\varphi||_{W^{1,2}(\Omega)} + ||f - L\varphi||_{L^2(\Omega)}\big) \quad \Rightarrow$$

$$||u||_{W^{2,2}(\Omega')} \leq C \cdot \big(||u||_{W^{1,2}(\Omega)} + ||f||_{L^2(\Omega)} + ||\varphi||_{W^{2,2}(\Omega)}\big),$$

since $\varphi \in W^{2,2}(\Omega)$ and L is of second order.

So we assume indeed $u \in W_0^{1,2}(\Omega)$. We now take a neighborhood containing a boundary portion and map it through a $\mathcal{C}^2$ diffeomorphism ψ (i.e ψ^{-1} exists and is $\mathcal{C}^2$) onto $\mathbb{R}^n$ with the boundary portion mapping into the hyperplane $\{x_n = 0\}$.

We pull back everything onto the flat boundary situation using $(\psi^{-1})^\star$– the original equation is

$$\forall \;\; v \in \mathcal{C}_0^1(\Omega) \qquad -\int_\Omega a^{ij}\mathrm{D}_j u \mathrm{D}_i v + \int_\Omega (b^i \mathrm{D}_i u + cu)v = \int_\Omega fv,$$

and the pulled-backed one

$$\forall \;\; v\circ\psi^{-1} \in \mathcal{C}_0^1(\psi(\Omega)) \qquad -\int_{\psi(\Omega)} \mathrm{Jac}(\psi^{-1}) a^{ij}\circ\psi^{-1}(\psi^{-1})^\star \mathrm{D}_j u (\psi^{-1})^\star \mathrm{D}_i v$$

$$+\int_{\psi(\Omega)} \mathrm{Jac}(\psi^{-1})(b^i\circ\psi^{-1}(\psi^{-1})^\star\mathrm{D}_i u + c\circ\psi^{-1})v\circ\psi^{-1} = \int_\Omega fv.$$

As we can assume ψ^{-1} preserves the given orientation of $\mathbb{R}^n$ and it is a diffeomorphism then $\mathrm{Jac}(\psi^{-1}) > 0$ and therefore we still have a strictly elliptic equation $\tilde{L}\tilde{u} = \tilde{f}$ and the $\mathcal{C}^2$ of ψ^{-1} guarantees the $\tilde{f} \in L^2(\Omega)$ and that $\tilde{a}^{ij}$ is still Lipschitz, and b^i and c still bounded (e.g. $(\psi^{-1})^\star(b^i\mathrm{D}_i u) = b_i\circ\psi^{-1}\cdot(\psi^{-1})^\star\mathrm{D}_i u = b_i\circ\psi^{-1}\cdot\mathrm{D}_i(\psi^{-1})^\star u = b_i\circ\psi^{-1}\cdot\mathrm{D}_i(u\circ\psi^{-1}) = b_i\circ\psi^{-1}\cdot\mathrm{D}_k u\cdot\mathrm{D}_i(\psi^{-1})_k$ (summation over k)).

Now we note that the difference quotients proof from last time still works for $\Delta_l^h\tilde{u}$ for each of the directions $l = 1,\ldots,n-1$ tangent to the boundary. So by applying that Theorem we

get $D_l\tilde{u} \in W^{1,2}(\psi(\Omega'))$ and hence $D_{ij}\tilde{u} \in L^2(\psi(\Omega'))$ except for $i = j = n$. Since ψ is a $\mathcal{C}^2$ diffeomorphism, the same holds for u.

So in order to finish the proof we go back to the proof. We have $W^{2,2}$ except possibly in the boundary, so may write the equation

$$Lu = a^{ij}D_ij u + D_i a^{ij} D_j u + b^i D_i u + c \cdot u = f,$$

a.e. All terms are in L^2 except $a^{nn}D_{nn}u$. But then isolating it on one side of the equation we see it must be as well, so it can not blow up at the boundary.

So now indeed we are done: we cover all of $\bar{\Omega}$ with a finite number of small ball cover the boundary portion and another Ω' covering the rest of the interior and we have the desired estimate on each of those domains. $\blacksquare$

We now have higher regularity upto the boundary:

Corollary. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = f$ in Ω, and $u = \varphi$ on $\partial\Omega$ (i.e $u - \varphi \in W_0^{1,2}(\Omega)$) and assume*

- *L strictly elliptic with $(a^{ij}) > \gamma \cdot I, \ \ \gamma > 0$*
- *$a^{ij} \in \mathcal{C}^{k,1}(\bar{\Omega})$*
- *$b^i, c \in \mathcal{C}^{k-1,1}(\bar{\Omega})$*
- *$f \in W^{k,2}(\Omega)$.*
- *$\partial\Omega$ is $\mathcal{C}^{k+2}$.*

Then $u \in W^{k+2,2}(\Omega)$ uniformly on the whole domain and

$$||u||_{W^{k+2,2}(\Omega)}$$

$$\leq C(||a^{ij}||_{C^{k,1}(\Omega)}, ||b||_{C^{k-1,1}(\Omega)}, ||c||_{C^{k-1,1}(\Omega)}, \lambda, \partial\Omega, \Omega', \Omega, k, n) \cdot \big(||u||_{W^{1,2}(\Omega)} + ||f||_{W^{k,2}(\Omega)} + ||\varphi||_{W^{k+2,2}(\Omega)}\big).$$

If $k = \infty$ then $u \in \mathcal{C}^\infty(\bar{\Omega})$.

The only difference from the compactly supported case is that we need now to have at our disposal a modified Sobolev Embedding: $W^{k+2,2} \subseteq \mathcal{C}^m(\bar{\Omega})$ instead of the one we proved with $W_0^{k+2,2} \subseteq \mathcal{C}^m(\bar{\Omega})$. This is indeed the case as one can show by modifying the latter's proof under the assumption of smooth enough boundary.

Improvement of our estimate

Assume $u \in W_0^{1,2}(\Omega)$, $Lu = f \in L^2(\Omega)$, $a^{ij}, b^i, c \in L^\infty(\Omega)$. *Then*

$$||u||_{W^{2,2}(\Omega)} \leq c \cdot \big(||u||_{L^2(\Omega)} + ||f||_{L^2(\Omega)}\big).$$

Proof. During the proof which involved the $W^{1,2}(\Omega)$ norm on the RHS we arrived at the inequality

$$\lambda||\mathrm{D}u||_{L^2(\Omega)} \leq \int_\Omega a^{ij}\mathrm{D}_i u \mathrm{D}_j \varphi = \int_\Omega (-b^i \mathrm{D}_i u - cu + f)\varphi$$

for all test functions $\varphi \in \mathcal{C}_0^1(\Omega)$ but in fact also for all test functions in the completion $W_0^{1,2}(\Omega)$! In particular we can take $\varphi = u$! But actually for $\varphi = u$ we can get this directly just from the strict ellipticity without having to go through the difference quotients process (just true for this special choice of v !). In particular we needn't assume more than $L^\infty(\Omega)$ regularity on the a^{ij} now! We continue then and get

$$\begin{aligned}\lambda||\mathrm{D}u||_{L^2(\Omega)} \leq \int_\Omega a^{ij}\mathrm{D}_i u \mathrm{D}_j u = \int_\Omega (-b^i \mathrm{D}_i u - cu + f)u \\ = -\int_\Omega (-b^i u(\mathrm{D}_i u) + \int_\Omega (-cu^2 + fu) \\ \leq \frac{1}{2}\epsilon \int_\Omega |\mathrm{D}u|^2 + \sum_{i=1}^n \frac{1}{2}\frac{1}{\epsilon}\int_\Omega |b^i u|^2 + \int_\Omega (-cu^2 + fu).\end{aligned}$$

For $\epsilon < \lambda$ one can move the first term to the LHS to conclude that (thanks to strict ellipticity we can now divide out by λ and get a uniform bound!)

$$||\mathrm{D}u||_{L^2(\Omega)} \leq \frac{1}{2}||u||_{L^2(\Omega)} + ||f||_{L^2(\Omega)}.$$

Now we can plug this in to our original estimate to get the desired improvement

$$\begin{aligned} ||u||_{W^{2,2}(\Omega)} &\leq c \cdot \big(||u||_{W^{2,1}(\Omega)} + ||f||_{L^2(\Omega)}\big). \\ &= c \cdot \big(||u||_{L^2(\Omega)} + ||\mathrm{D}u||_{L^2(\Omega)} + ||f||_{L^2(\Omega)}\big). \\ &\leq c' \cdot \big(||u||_{L^2(\Omega)} + ||f||_{L^2(\Omega)}\big). \qquad \blacksquare \end{aligned}$$

Similarly this improvement applies to $u \in W^{1,2}(\Omega)$ though it will not apply up to the boundary; we will have

$$||u||_{W^{1,2}(\Omega')} \leq c\big(||u||_{L^2(\Omega)} + ||f||_{L^2(\Omega)}\big)$$

by taking $\varphi = \eta \cdot u$ with $\eta = 1$ on Ω' and applying the above argument. If we want an estimate on all of Ω we need to add a term $||\varphi||_{W^{2,2}(\Omega)}$ to the RHS by applying the above result to $u - \varphi \in W_0^{1,2}(\Omega)$ for $f' := f - L\varphi$.